Integral geometry problems on the family of broken lines and semicircles

Seidullaev A.K.

Karakalpak State University named after Berdakh

Table of Contents

Introduction

Integral geometry has a wide range of applications in various fields, including medical tomography, geophysics, computer vision, materials science, and others. Reconstruction of functions from integral characteristics makes it possible to obtain information about the internal structures of objects that may not be available by direct observation or measurement. This opens new possibilities for analysis and diagnostics, allows to detect hidden properties of objects and reduces the need for destructive or invasive research methods.

- M. M. Lavrentiev, A. L. Bukhheim. About one class of operator equations of the first kind // Functional Analysis and its Applications, 1973, 7(4), pp.44-53. (in Russian)
- A. L. Buchheim. About some problems of integral geometry // Sib.mat.zhurn., 1972, 13(1), pp.34-42. (in Russian)
- V. G. Romanov. On some classes of singularity of solvability of integral geometry problems // Mat. notes, 1974, 16(4), pp.657-668. (in Russian)

Introduction

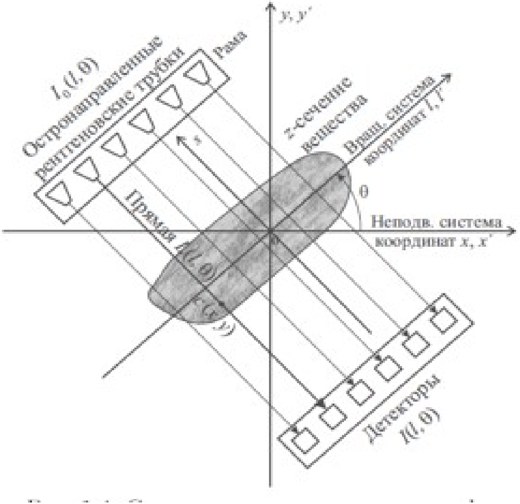


Рис.: Schematic diagram of the X-ray tomograph

Introduction

Let us introduce a coordinate system $x,\ y$ fixed with respect to the object and a rotating coordinate system $l,\ s$ associated with the frame. According to Bera's law[1], the intensity of the X-ray beam received by some detector is equal to

$$I\left(l,\theta\right)=I_{0}\left(l,\theta\right)e^{-\int\limits_{L(l,\theta)}c(x,y)ds}, \qquad (1)$$

where l is the coordinate of the detector, θ is the angle of rotation of the frame, $I_0(l,\theta)$ is the intensity of the corresponding emitting tube (usually $I_0 = const$), $L(l,\theta)$ is the line defined by the equation: $x\cos\theta + y\sin\theta = l$. $c(x,y)$ is the density of matter (more precisely, the X-ray absorption coefficient of matter) on the beam $L(l,\theta)$.

[1] Sivukhin D. V. Absorption of light and broadening of spectral lines // General course of physics. - M., 2005. - Vol. IV. Optics. - P. 582-583

Let us rewrite (1) in a different way

$$\int_{L(l,\theta)} c(x,y)ds = q(l,\theta). \tag{2}$$

Radon obtained one of the solutions of equation (2), which is written as

$$c\left(x,y\right) = -\frac{1}{2\pi^2}\int\limits_0^{\pi} d\theta \int\limits_{-\infty}^{+\infty} \frac{\partial q\left(l,\theta\right)}{\partial l}\frac{dl}{l-\left(x\cos\theta + y\sin\theta\right)}.$$

Introduction

The most important stage of any tomographic experiment is the reconstruction of the object structure from the obtained projections. In the mathematical sense, this represents a general problem of integral geometry, which is usually formulated as follows[2]:
Let $u(x)$ and $g(x,y)$ be sufficiently smooth functions defined in n -dimensional and $n+k$ -dimensional spaces, respectively, $x=(x_1,x_2,....,x_n)$ and $y=(y_1,y_2,...,y_k)$ vectors, $M(y)$ is some family of smooth manifolds. Known integrals

$$\int\limits_{M(y)} g(x,y)\,u(x)\,d\sigma = f(y),$$

and weight functions $g(x,y)$, where $d\sigma$ is an element of the measure on $M(y)$. We need to find $u(x)$.

[2] M.M. Lavrentev, V.G. Romanov, S.P. Shishatskii, Ill-posed problems of mathematical physics and analysis, Providence, American Mathematical Society, 1986, p. 290.

Statement of the problem of integral geometry on the family of broken lines

Determine the functions $u(x,y)$ in a given strip

$$\Omega = \left\{(x,y) : x \in R^1, 0 \le y \le H < \infty\right\},$$

by computing its integrals over the broken lines belonging to the family of

$$\Gamma(x,y,\varphi) = \left\{(\xi,\eta) :\ |x-\xi| = (y-\eta)\, tg\varphi, \varphi \in \left(0, \frac{\pi}{2}\right)\right\}$$

with a given weight function $g(x,\xi) = (x-\xi)\, e^{-k(y-\eta)},\ k \ge 0$:

$$\int\limits_{\Gamma(x,y,\varphi)} g(x,\xi)\, u(\xi,\eta)\, d\xi \ = f^{\delta}(x,y)\,. \qquad (3)$$

The right-hand side of equation (3) is represented as an approximation, i.e., $\left\|f^{\delta} - f\right\|_{L_2} \le \delta$, where δ – represents the upper bound of the right-hand side of (3).

Inversion formula

Theorem 1.

Let the function $f^{\delta}(x,y)$ be known for all $(x,y)\in\Omega$. Then the solution of equation (3) in the class $C_0^2(\Omega)$ is unique and is expressed through the function by the formula

$$u(x,y)=\frac{\cos\varphi}{2}\int\limits_{-\infty}^{x}\left(tg^2\varphi\frac{\partial^2}{\partial\xi^2}f^{\delta}(\xi,y)-2k\frac{\partial}{\partial y}f^{\delta}(\xi,y)-k^2f^{\delta}(\xi,y)-\frac{\partial^2}{\partial y^2}f^{\delta}(\xi,y)\right)d\xi. \quad (4)$$

For the case $k = 0, \ \varphi = 45^0$ the solution of equation (4) has the following analytical inversion formula in the form of

$$u(x,y) = \frac{1}{2\sqrt{2}} \int\limits_a^x \left(\frac{\partial^2}{\partial \xi^2} f^\delta(\xi, y) - \frac{\partial^2}{\partial y^2} f^\delta(\xi, y) \right) d\xi. \quad (5)$$

Let's denote:

$$\psi(x,y) = \frac{\partial^2}{\partial x^2} f^\delta(x,y), \quad (6)$$

$$\phi(x,y) = \frac{\partial^2}{\partial y^2} f^\delta(x,y). \quad (7)$$

Let's introduce a regular grid in the rectangular region $D = [a, \, b] \times [0, \, d]$. Let us rewrite (6) in the form

$$A\psi = \int\limits_a^x K_1(x,s)\, \psi(\cdot, s) ds = \tilde{f}^\delta(x, \cdot)$$

где $\tilde{f}^\delta(x,\cdot) = f^\delta(x,\cdot) - f^\delta(a,\cdot), \ K_1(x,s) = x - s.$

In order to ensure stability of the solution of the last equation, the condition of minimum of the smoothing functional is applied

$$\Phi_{\alpha_x}[\psi_{\alpha_x}, f^\delta] = \left\| A\psi_{\alpha_x} - \tilde{f}^\delta(x,\cdot) \right\|_{L_2} + \alpha_x \|\psi_{\alpha_x}\|_{L_2}, \ \alpha_x > 0. \tag{8}$$

By expanding (8) we have the following equation of the second kind:

$$\alpha_x \psi_{\alpha_x}(t,\cdot) + \int_a^b R(t,s)\,\psi_{\alpha_x}(s,\cdot)\,ds = \int_t^b (x-t)\,\tilde{f}^\delta(x,\cdot)\,dx, \tag{9}$$

where $R(t,s) = \int\limits_{\max\{t,s\}}^{b} K_1(x,s)\,K_1(x,t)dx.$

We discretise equation (9) on a regular grid. As a result, we obtain a system of algebraic linear equation of the form

$$\alpha_x \psi_i + \sum_{k=1}^{n_x} R(t_i, s_k) h_k \psi_i = Q_i, \tag{10}$$

$$Q_i = \int_{t_i}^{b} (x - t) \tilde{f}^{\delta}(x, \cdot) \, dx,$$

where $i = \overline{1, n_x}$, $h_1 = h_{n_x} = \frac{h_x}{2}$, $h_k = h_x$, $k = \overline{2, n_x - 1}$.
Let M be a matrix with elements $M_{ik} = R(t_i, s_k) h_k$. Then the system of equations (10) with respect to the vector ψ with components $(\psi_1, \psi_2, ..., \psi_{n_y})$ can be written in the form

$$M_{\alpha_x} \psi_{\alpha_x} \equiv M \psi_{\alpha_x} + \alpha_x E \psi_{\alpha_x} = Q, \tag{11}$$

$$\psi_{\alpha_x} = (M + \alpha_x E)^{-1} \cdot Q,$$

where Q is a vector with components $(q_1/2, q_2, ..., q_{n_x})$ and E is a unitary matrix.

Similarly, solving according to the scheme described above, we find solutions to equation (7). For $\alpha_x \to 0$ and $\alpha_y \to 0$, solutions to equation (5) can be found by numerically integrating the following integral using the trapezoidal formula

$$u\left(x_i, y_j\right) = \frac{1}{2\sqrt{2}} \int\limits_a^{x_i} \Big(\psi_{\alpha_x^{opt}}\left(\xi, y_j\right) - \phi_{\alpha_y^{opt}}\left(\xi, y_j\right)\Big) d\xi. \tag{12}$$

NUMERICAL EXPERIMENT

Let us consider in a numerical experiment the Shepp-Logan model phantom.

Ellipse	The centre of the ellipse	Semi-major axis	Semi-minor axis	Rotation angle	ρ
1	(0,0)	0.69	0.92	0	2
2	(0,0.0184)	0.6624	0.874	0	0.98
3	(0.22,0)	0.11	0.31	18°	0.02
4	(0.22,0)	0.16	0.41	8°	0.02
5	(0,0.35)	0.21	0.25	0	0.01
6	(0,0.1)	0.046	0.046	0	0.01
7	(0,0.1)	0.046	0.046	0	0.01
8	(0.08,0.605)	0.046	0.023	0	0.01
9	(0,0.605)	0.023	0.023	0	0.01
10	(0.06,0.605)	0.023	0.046	0	0.01

Рис.: The phantom of Shepp-Logan

In the numerical experiment we will use the standard normal distribution [3]
$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)}$ i.e., when $\sigma = 1, \quad m = 0.$

$$f^{\delta}(x, \cdot) = f(x, \cdot) + p \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},$$

$$\sigma(\alpha_x) = \frac{\|\psi_{\alpha_x}(x, \cdot) - \psi(x, \cdot)\|}{\|\psi(x, \cdot)\|}.$$

[3] Ventzel E. S. Probability Theory. - 10th ed., stereotype. - Moscow: Academia, 2005. - 576 c. - ISBN 5-7695-2311-5

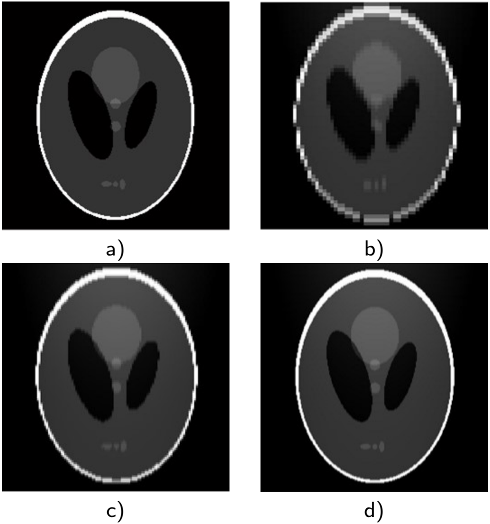


Таблица: Reconstruction of Sheep-Logan phantom. a) Original phantom b) reconstruction at N=64x64 with 5% noise, c) reconstruction at N=128x128 with 5% noise, d) reconstruction at N=256x256 with 5% noise.

Comparison with Hann and Cosine methods

Method	N value	Error analysis
Hann	64	0,6630
	128	0,5454
	256	0,4022
Cosine	64	0,6836
	128	0,5619
	256	0,4120
Tikhonov regularisation	64	0,4402
	128	0,3866
	256	0,3656

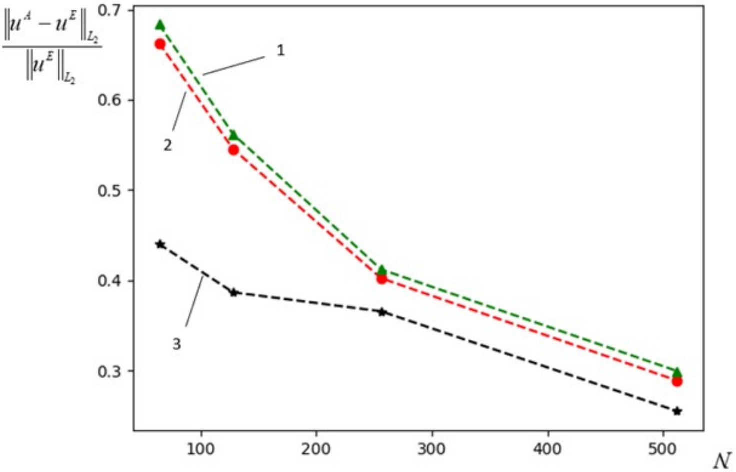


Рис.: Comparison. 1-Method of Cosine, 2- Hann, 3-Tikhonov regularisation

Statement of the IGP on the family of semicircles

Let us consider a mathematical model for determining the internal structure of objects on a family of semicircles $\{\Upsilon_3\,(x,\,y)\}$:

$$\int\limits_{\Upsilon_3(x,y)} u\,(\xi,\eta)\,d\xi = f^{\delta}\,(x,y), \tag{13}$$

where an arbitrary curve of the family is represented as
$\Upsilon_3\,(x,y) = \Big\{(\xi,\eta):\ (x-\xi)^2 + \eta^2 = y^2,\ 0 \le \eta \le y \le H,\ -\infty < x < \infty\Big\}.$

Lemma

Lemma

Let $f^{\delta}(x,y)$ be a twice continuously differentiable function on arguments and finite in the strip Ω. Then the solution of equation (13) in the class of twice continuously differentiable finite functions with a carrier in the strip Ω is singular and the representation takes place

$$\hat{u}(\lambda,y)=\frac{1}{\pi y}\frac{\partial}{\partial y}\int\limits_{0}^{y}\frac{\eta ch\left(\lambda\sqrt{y^{2}-\eta^{2}}\right)}{\sqrt{y^{2}-\eta^{2}}}\cdot\hat{f}^{\delta}(\lambda,\eta)\,d\eta.$$

Theorem 2

Consider the operator

$$R(u,\alpha) = L^{-1}\left[\psi(u(\lambda),\lambda)\cdot R_{\alpha_1}(\lambda,\alpha_1)\right],$$

where L^{-1} is the inverse Fourier transform, and $R_{\alpha_1}(\lambda,\alpha_1) = e^{-\alpha_1^2\lambda^2}$ is some given function, defined for all non-negative values of the parameter α and any λ for which the inverse Fourier transform is taken.

Theorem 2.

Let the function $f(x,y)$ be given in the strip Ω. Then the approximate solution $u_\alpha(x,y)$ of equation (13) in the class of twice continuously differentiable finite functions with a carrier in the strip Ω has the following form

$$u_\alpha(x,y) = \frac{1}{2\alpha_1 y\sqrt{\pi^3}}\frac{\partial}{\partial y}\int_0^y\int_{-\infty}^{+\infty} e^{\frac{y^2-\eta^2-(x-\xi)^2}{4\alpha_1^2}}\cos\frac{(x-\xi)\sqrt{y^2-\eta^2}}{2\alpha_1^2}\frac{\eta f^\delta(\xi,\eta)}{\sqrt{y^2-\eta^2}}d\xi d\eta.$$

The scheme of the algorithm for solving the problem is as follows:
Step 1: We divide the segment $[a; b]$ on the Ox-axis and $[c; d]$ on the Oy-axis into $n_x - 1$ and $n_y - 1$ parts, respectively, i.e. $x_i = a + (i-1)\,h_x,\ \ y_j = c + (j-1)\,h_y$.
Step 2: The approximations of the functions $u\,(x_i,\, y_j)$ will be denoted by u_{ij}^A.

$$u_{\alpha_1}\,(x_i, y_j) = \frac{G_{ij+1} - G_{ij-1}}{4\alpha_1 y_j h_y \sqrt{\pi^3}}, \tag{14}$$

$$G_{ij} = \int_0^{y_j}\int_{-1}^{+1} e^{\frac{y_j^2 - \eta^2 - (x_i - \xi)^2}{4\alpha_1^2}} \cos\frac{(x_i - \xi)\sqrt{y_j^2 - \eta^2}}{2\alpha_1^2}\,\frac{\eta f^{\delta}\,(\xi, \eta)}{\sqrt{y_j^2 - \eta^2}}\,d\xi d\eta,$$

where α is the regularisation parameter.

Constructions of approximate solutions on the family of semicircles

Let us consider another way of constructing the stabilising multiplier. Let $M(\lambda) = 1 + \lambda^2$ be a given even function, and:

1. $M(\lambda)$ is piecewise continuous on any finite segment;
2. $M(0) \geq 0$ and $M(\lambda) > 0$ at $\lambda \neq 0$;
3. For sufficiently large $|\lambda|$: $M(\lambda) \geq C > 0$
4. For any $\alpha_2 > 0$

$$\frac{ch\left(\lambda\sqrt{y^2-\eta^2}\right)}{ch^2\left(\lambda\sqrt{y^2-\eta^2}\right) + \alpha_2\left(1+\lambda^2\right)} \in L_2(-\infty, \infty).$$

Theorem 3

Let the function $f(x, y)$ be defined in the strip Ω. Consider the regularising operator for equation (13)

$$R_f(u, \alpha) = \mathcal{F}^{-1}\left[\hat{u}(\lambda, y) \cdot R_{\alpha_2}(\lambda, \alpha_2)\right]$$

Then the solution of equation (13) in the class of twice continuously differentiable finite functions with a carrier in the strip Ω has the following form

$$u_{\alpha_2}(x, y) = \frac{1}{\pi y}\frac{\partial}{\partial y}\int_0^y\int_{-\infty}^{+\infty} \frac{ch^2\left(\lambda\sqrt{y^2-\eta^2}\right)}{ch^2\left(\lambda\sqrt{y^2-\eta^2}\right) + \alpha_2\left(1+\lambda^2\right)} \cdot \frac{\eta \hat{f}^{\delta}(\lambda, \eta)\, e^{-i\lambda x}}{\sqrt{y^2-\eta^2}} d\lambda d\eta.$$

For the numerical experiment we introduce regular meshes by x, y, $x \in [-a,\, a]$, $y \in \; [c, d]$, $x_k = -a + kh_x, \;\; y_j = c + jh_y$, где $h_x = \frac{2a}{n_x}$, $h_y = \frac{d-c}{n_y}$.
The discrete Fourier transform for the function $f\,(x_i, \cdot)$ is defined by the formula

$$\hat{f}\,(\lambda_m, \cdot) = \sum_{k=0}^{n_x-1} f\,(x_k, \cdot) e^{i\lambda_m x} = \sum_{k=0}^{n_x-1} f\,(x_k, \cdot) e^{i2\pi km/n_x}, \;\; m = 0, 1, ..., n_x - 1,$$

here $m = m\Delta\lambda, \;\; \Delta\lambda = \frac{2\pi}{T}$.
Inverse discrete Fourier transform

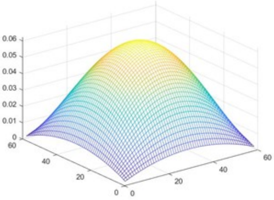
$$f\,(x_k, \cdot) = \sum_{m=0}^{n_x-1} \hat{f}\,(\lambda_m, \cdot) e^{-i2\pi km/n_x}, \;\; k = 0, 1, ..., n_x - 1.$$

Taking into account the above mentioned, let us rewrite the inversion formula in the following form

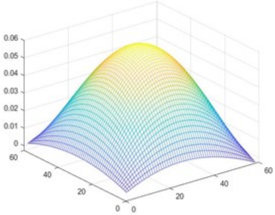
$$u_{\alpha_2}\,(x_k, y_j) = \frac{1}{\pi y} \frac{\partial}{\partial y} \int\limits_0^{y_j} \frac{T\,(x_k, y_j, \eta)\, d\eta}{\sqrt{y_j^2 - \eta^2}}, \tag{15}$$

where $T\,(x_k, y_j, \eta) = \int\limits_{-a}^{+a} \frac{ch^2\left(\lambda\sqrt{y_j^2-\eta^2}\right)\;\eta \hat{f}(\lambda,\eta) e^{-i\lambda x_k}}{ch^2\left(\lambda\sqrt{y_j^2-\eta^2}\right)+\alpha_2\left(1+\lambda^2\right)} d\lambda.$

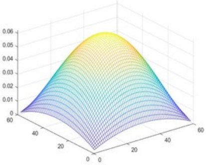
$$f\left(x,y\right)=\frac{1}{240}y^{2}\left(-15\pi\left(-4+4x^{2}+y^{2}\right)+64y\left(-5+5x^{2}+y^{2}\right)\right)$$



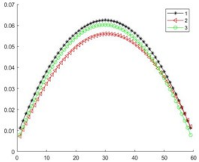
a)



b)



c)



d)

Таблица: Results of test function recovery. a) exact solution, b) recovery by formula (14) c) recovery by formula (15) d) comparison of recovery results 1 - exact solution, 2 - recovery by (14), 3 - recovery by (15).

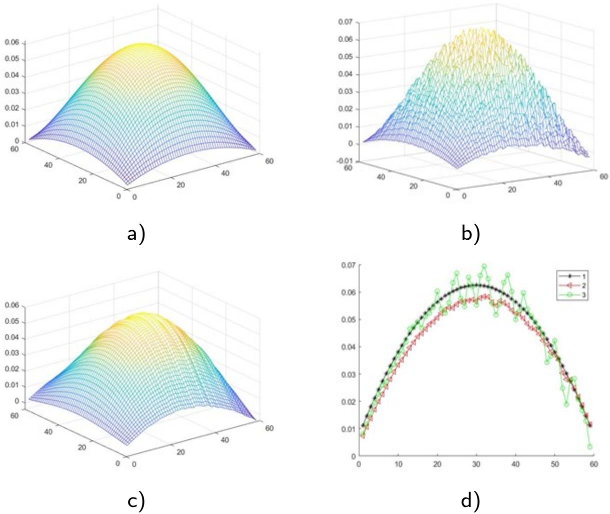


Таблица: Results of test function reconstruction. a) exact solution, b) reconstruction by formula (15), noise 5% c) reconstruction by formula (14), noise 5% d) comparison of results by cross-section of the reconstructed solution 1) exact solution, 2) reconstruction by formula (14), 3) reconstruction by formula (15).

N	Relative error on $R_{\alpha_1}(\hat{u}, \alpha_1) = e^{-\alpha_1^2 \lambda^2}$ without noise	Relative error on $R_{\alpha_2}(\hat{u}, \alpha_2)$ without noise	Relative error on $R_{\alpha_1}(\hat{u}, \alpha_1) = e^{-\alpha_1^2 \lambda^2}$ with noise 5%	Relative error on $R_{\alpha_2}(\hat{u}, \alpha_2)$ with noise 5%
32	0,0766693	0,0767512	0,079985	0,0796571
64	0,0509354	0,0714749	0,0683423	0,0973186

Thank you!